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On the Time Optimal Course Changing of Ships

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SUMMARY
The mathematical theory of optimal control is applied to the problem of steering a ship as quickly as possible to a new course. In the linearized theory such an optimal control exists for each change of course and will be discussed for stable as well as unstable ships.

1. Introduction

A ship follows a certain rectilinear course and wants to change to another one. The initial course and the final one are prescribed. The problem we will discuss is how to change the rudder angle with time to realize the prescribed change of course as quickly as possible. It is assumed that the forward speed of the ship is constant.

In order to define the mechanical behaviour of the ship we need its equations of motion [1], which contain the rudder angle δ, confined between two values, as a control variable. We will consider the linearized equations. The coefficients of these have been determined in an experimental way [5], [6].

To solve this steering problem a theory of trajectory optimization is needed. By means of the "maximum principle" of Pontryagin [8], a geometrical insight into possible optimal trajectories of the ship is obtained. The desired optimal trajectory cannot be found in a straightforward manner. Several trials must be made and the prescribed change of course has to be found by interpolation. This method is practically limited to linear time invariant systems of second or third order. In our case we have a third order problem.

In the last section of this paper another method of solving the problem is discussed briefly. This method is also suitable for higher order systems.

We assume that the rudder can change its angle instantaneously. Then as a consequence of the application of the maximum principle to our equations of motion the rudder angle will be a piecewise constant function of time. In practice this can never be realized, but for a not too small angle of course changing the piecewise constancy of the rudder angle forms a good approximation.

Our results are compared with those obtained by employing still more simplified equations of motion, developed by Nomoto [7]. It appears that the results agree rather well for not too small changes of course.

This paper is an extract of [9].

2. The Equations of Motion of a Ship

It is assumed that the ship moves through undisturbed water and air. Moreover we consider only its motion in the horizontal plane, hence for instance pitching and rolling are neglected. The number of degrees of freedom possessed by the ship is therefore limited to three: motion of the center of gravity in the two dimensions of the horizontal plane and an angular orientation with respect to a fixed direction. We use a Cartesian coordinate system fixed to the ship. The origin of the system coincides with the center of gravity, the x-axis coincides with the axis of symmetry of the ship and the y-axis is perpendicular to this axis. This choice has the advantage that the force required to accelerate the water surrounding the ship can be put into a particularly simple form [3].

The following quantities, partly denoted in Figure 2.1, are introduced:

V velocity of the ship.
v, η components of V in x and y direction.
ψ driftangle.
θ angular position of the x-axis with respect to the reference axis l.
δ rudder angle.
L length of the ship.
ρ specific density of water.
t time.
r angular velocity, i.e. $r = d\theta/dt$.
m mass of the ship.
I moment of inertial taken around the vertical axis through the center of gravity.

Arrows indicate the directions in which the quantities are positive.

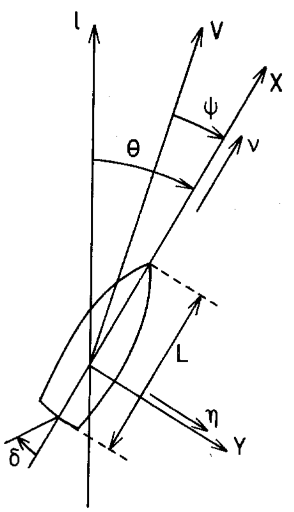

Figure 2.1 Coordinates and velocities defining the state of the ship.

The equations of motion are [1], [3]:

$$\dot{\theta} = r, \tag{2.1}$$

$$m(\dot{v} - r \cdot \eta) = X, \tag{2.2}$$

$$m(\dot{\eta} + r \cdot v) = Y, \tag{2.3}$$

$$I\dot{r} = N, \tag{2.4}$$

where X, Y are the forces and N the moment about the center of gravity, acting on the ship. A dot above a symbol denotes its derivative with respect to time. In general X, Y and N depend on v, η, r, δ, $\dot{v}$, $\dot{\eta}$ and $\dot{r}$. In this paper it is assumed that the forward speed is constant and will be denoted by v_0. Note that by constant power input to the screw the speed of advance will decrease by manoeuvring, so that in case of a constant forward speed the power input must be properly adjusted. In addition, because we will only consider the linearized theory, the term $-m \cdot r \cdot \eta$ in equation (2.2) is neglected and because the forward speed is constant X becomes zero. This means that in the x-direction the thrust always equals the resistance of the ship in water and air. Equation (2.2) becomes irrelevant.

The ship is said to be in equilibrium if $r = \eta = 0$. Because we consider only small disturbances of the equilibrium position, Y and N are assumed to be linear functions of η, r, $\dot{\eta}$, $\dot{r}$ and δ;

$$Y = Y_\eta \cdot \eta + Y_r \cdot \eta + Y_\delta \cdot \delta + Y_{\dot\eta} \cdot \dot\eta + Y_{\dot r} \cdot \dot r, \tag{2.5}$$

$$N = N_\eta \cdot \eta + N_r \cdot r + N_\delta \cdot \delta + N_{\dot\eta} \cdot \dot\eta + N_{\dot r} \cdot \dot r. \tag{2.6}$$

The quantities m, I, Y_η, $Y_{\dot\eta}$, Y_r, $Y_{\dot r}$, Y_δ, N_η, $N_{\dot\eta}$, N_r, $N_{\dot r}$ and N_δ are constants which have been measured for several types of ship models [5], [6]. Substitution of equations (2.5) and (2.6) in (2.3) and (2.4) yields

$$(m - Y_{\dot\eta}) \cdot \dot\eta - Y_{\dot r} \cdot \dot r = Y_\eta \cdot \eta + (Y_r - m \cdot v) \cdot r + Y_\delta \cdot \delta\,, \tag{2.7}$$

$$-N_{\dot\eta} \cdot \dot\eta + (I - N_{\dot r}) \cdot \dot r = N_\eta \cdot \eta + N_r \cdot r + N_\delta \cdot \delta\,. \tag{2.8}$$

At this point we introduce dimensionless quantities.

These new quantities, provided with a bar, are obtained from the quantities with dimensions in the following manner:

$$\bar v = \frac{v}{v_0} = 1\,; \quad \bar\eta = \frac{\eta}{v_0}\,; \quad \bar r = \frac{rL}{v_0}\,; \quad \bar{\dot\eta} = \frac{\dot\eta L}{v_0^2}\,;$$
$$\bar{\dot r} = \frac{\dot r L^2}{v_0^2}\,; \quad \bar m = \frac{m}{\frac12\rho\cdot L^3}\,; \quad \bar I = \frac{I}{\frac12\rho\cdot L^5}\,; \quad \bar Y_\eta = \frac{Y_\eta}{\frac12\rho\cdot v_0\cdot L^2}\,;$$
$$\bar Y_{\dot\eta} = \frac{Y_{\dot\eta}}{\frac12\rho\cdot L^3}\,; \quad \bar Y_r = \frac{Y_\eta}{\frac12\rho\cdot v_0\cdot L^3}\,; \quad \bar Y_{\dot r} = \frac{Y_{\dot r}}{\frac12\rho\cdot L^4}\,; \quad \bar Y_\delta = \frac{Y_\delta}{\frac12\rho\cdot v_0^2\cdot L^2}\,;$$
$$\bar N_\eta = \frac{N_\eta}{\frac12\rho\cdot v_0\cdot L^3}\,; \quad \bar N_{\dot\eta} = \frac{N_{\dot\eta}}{\frac12\rho\cdot L^4}\,; \quad \bar N_r = \frac{N_r}{\frac12\rho\cdot v_0\cdot L^4}\,; \quad \bar N_{\dot r} = \frac{N_{\dot r}}{\frac12\rho\cdot L^5}\,;$$
$$\bar N_\delta = \frac{N_\delta}{\frac12\rho\cdot v_0^2\cdot L^3}\,; \quad \bar t = \frac{t v_0}{L}\,. \tag{2.9}$$

Having introduced the dimensionless quantities, we omit the bars and equations (2.1), (2.7) and (2.8) can be written as

$$\frac{d\theta}{dt} = r\,, \tag{2.10}$$

$$\frac{d\eta}{dt} = b_3\eta + b_4 r + b_5\delta\,, \tag{2.11}$$

$$\frac{dr}{dt} = c_3\eta + c_4 r + c_5\delta\,, \tag{2.12}$$

where b_i, c_i, $i=3, 4, 5$ are constants which have been calculated from the (dimensionless) coefficients in equations (2.7) and (2.8).

If

$$D = \begin{vmatrix} m - Y_{\dot\eta} & -Y_{\dot r} \\ -N_{\dot\eta} & I - N_{\dot r} \end{vmatrix}, \tag{2.13}$$

then

$$b_3 = \frac1D \begin{vmatrix} Y_\eta & -Y_{\dot r} \\ N_\eta & I - N_{\dot r} \end{vmatrix}; \quad b_4 = \frac1D \begin{vmatrix} Y_r - m & -Y_{\dot r} \\ N_r & I - N_{\dot r} \end{vmatrix};$$
$$b_5 = \frac1D \begin{vmatrix} Y_\delta & -Y_{\dot r} \\ N_\delta & I - N_{\dot r} \end{vmatrix}; \quad c_3 = \frac1D \begin{vmatrix} m - Y_{\dot\eta} & Y_\eta \\ -N_{\dot\eta} & N_\eta \end{vmatrix}; \tag{2.14}$$
$$c_4 = \frac1D \begin{vmatrix} m - Y_{\dot\eta} & Y_r - m \\ -N_{\dot\eta} & N_r \end{vmatrix}; \quad c_5 = \frac1D \begin{vmatrix} m - Y_{\dot\eta} & Y_\delta \\ -N_{\dot\eta} & N_\delta \end{vmatrix}.$$

3. The Statement of the Problem

We want to change the course of the ship. Therefore we are only interested in the three coordinates θ, η and r. The initial situation will be $(\theta, \eta, r) = (0, 0, 0)$ and the prescribed final situation $(\gamma, 0, 0)$. The three quantities θ, η and r are described by the equations (2.10)–(2.12). We consider the motion of the ship in a three dimensional phase space, spanned by the θ-, η- and r-axes. In Figure 3.1 use is already made of results which will be discussed in section 4.

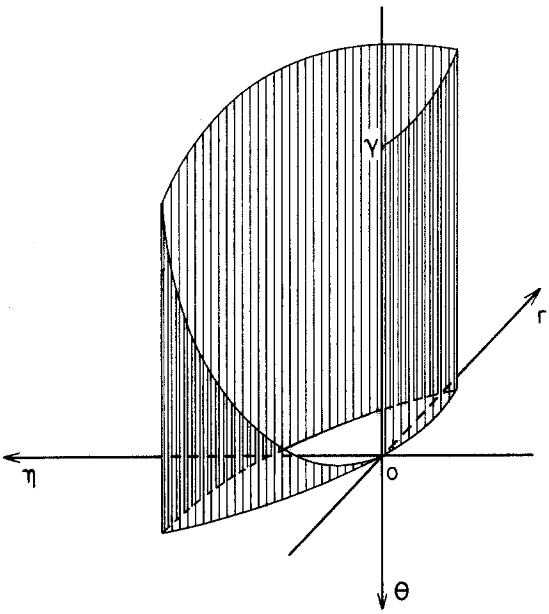

Figure 3.1 The motion of the ship in the (θ, η, r) space, with the projection of the trajectory in the (η, r) plane.

The question is how to reach the final position as quickly as possible from the initial position by steering the ship. So we have to determine the rudder angle δ as a function of time, $\delta = \delta(t)$. The rudder angle can be varied free within a previously given angular region $|\delta| \leqq \delta_{max}$. In our linearized theory we assume that δ_{max} is sufficiently small. From experiments [5] it can be concluded that the linearized theory is rather accurate with respect to δ for values of δ_{max} smaller than 20 degrees.

4. Application of Results of Control Theory

Equations (2.10)–(2.12) can be written in vectorform as

$$\frac{dx}{dt} = Ax + b\delta\,, \tag{4.1}$$

where

$$x = \begin{pmatrix} \theta \\ \eta \\ r \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ b_5 \\ c_5 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & b_3 & b_4 \\ 0 & c_3 & c_4 \end{pmatrix}.$$

It is easily seen that system (4.1) is controllable. For a definition of controllable systems see [4]. We now mention a known property of controllable linear processes on the assumption that the eigenvalues of matrix A are real and that $|\delta(t)| \leqq \delta_{max}$. The property in this case is, that each time optimal control $\delta(t)$ is piecewise constant, that it takes on only the values $+\delta_{max}$ and $-\delta_{max}$ and does not have more than $(n-1)$ switches, i.e. discontinuity points, where n is the order of system (4.1).

The eigenvalues of

$$\begin{pmatrix} b_3 & b_4 \\ c_3 & c_4 \end{pmatrix}$$

are denoted by α and β and these are also eigenvalues of A. The third eigenvalue of A is zero. We suppose that α and β are real and distinct. For all known types of ships α and β are real. So we know that for our problem at most two switches of the rudder exist.

It is clear that in practice "jumps" of the rudder can never be realized. But if we restrict ourselves to shipmanoeuvres which vary slowly with respect to the time interval needed for turning the rudder from $-\delta_{max}$ to $+\delta_{max}$ or vice versa, the piecewise continuity of $\delta(t)$ is a good approximation. This happens when the change of course is sufficiently large.

We assume that at a switch at time τ

$$\theta(\tau^-)=\theta(\tau^+), \quad \eta(\tau^-)=\eta(\tau^+) \text{ and } r(\tau^-)=r(\tau^+), \tag{4.2}$$

where τ^- denotes the moment directly before the switch and τ^+ the moment after. This is also a logical consequence of the equations of motion (2.10)–(2.12).

Because always $|\delta|=\delta_{max}$, the solution of our problem is composed of two types of trajectories of the form

$$\begin{aligned} \theta &= \int^t r\,dt \\ \eta_\pm &= C e^{\alpha t} + D e^{\beta t} \pm P \\ r_\pm &= M_1 C e^{\alpha t} + M_2 D e^{\beta t} \pm Q \end{aligned} \tag{4.3}$$

which are the solutions of equation (4.1) with $\delta = \pm\delta_{max}$; C and D are constants of integrations. The following abbreviations have been used:

$$\begin{aligned} P &= \delta_{max}\frac{b_4c_5-c_4b_5}{b_3c_4-c_3b_4}, \quad Q=\delta_{max}\frac{b_5c_3-b_3c_5}{b_3c_4-c_3b_4}, \\ M_1 &= \frac{\alpha-b_3}{b_4}, \quad M_2 = \frac{\beta-b_3}{b_4}. \end{aligned} \tag{4.4}$$

For the further description of optimal solutions we distinguish two cases. Firstly, $\alpha<0$ and $\beta<0$, and secondly $\alpha>0$ and $\beta<0$, or $\alpha<0$ and $\beta>0$. The third possibility, $\alpha>0$ and $\beta>0$, is only of theoretical interest and will not be discussed here ([9]).

5. Stable Ships; ($\alpha<0$, $\beta<0$)

Considering the two-dimensional (η, r) plane, we can sketch qualitatively the trajectories $(\eta_+(t), r_+(t))$ and $(\eta_-(t), r_-(t))$ (see Figure 5.1a and b). The trajectories have a node at the point $N_\pm$. The coordinates of $N_\pm$ are $(\pm P, \pm Q)$. The points N_+ and N_- are singular points of the differential equations, consisting of the second and third row of equation (4.1) with $\delta=\delta_{max}$ and $\delta=-\delta_{max}$ respectively.

In the (η, r) plane the initial and final position have been settled in the origin, so a closed curve must be constructed, running from the origin to the origin and consisting of parts of (η_+, r_+) and (η_-, r_-). Provisionally we do not take notice of θ. If we suppose that the ship starts with $\delta=+\delta_{max}$ at $t=0$, the first part of the trajectory is a part of the solution (η_+, r_+), in Figure 5.2 called l, passing through the origin. Along this solution the ship approaches N_+ for $t\to\infty$ and will never return to the origin. Therefore we need at least one switch of the rudder. Suppose this occurs at some $t=\tau_1>0$. The second part of the trajectory is a part of the solution (η_-, r_-), which passes through $(\eta_+(\tau_1), r_+(\tau_1))$. Along this solution the ship runs to N_- for $t\to\infty$. This part does not pass through the origin, as appears from equation (4.3). So we need yet another

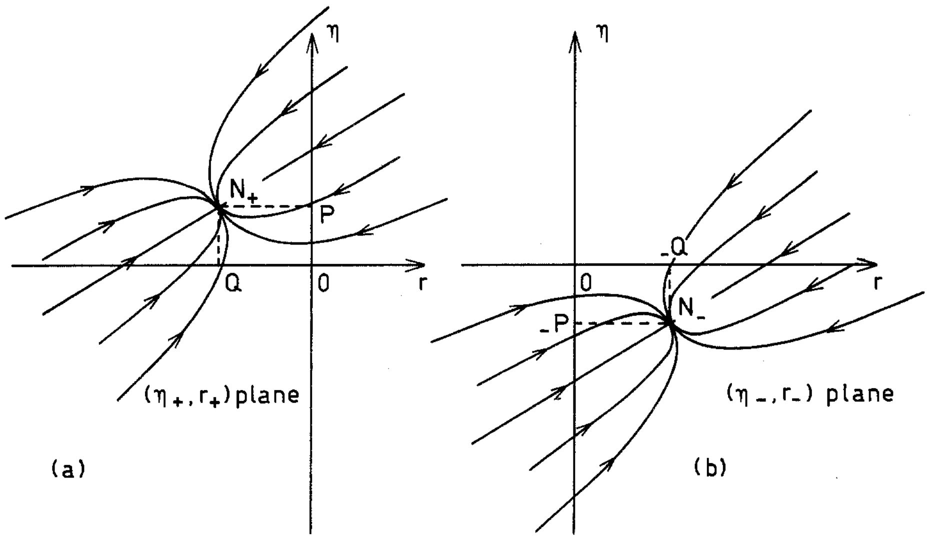

Figure 5.1 The trajectories $(\eta_\pm, r_\pm)$ in the (η, r) plane, in the case $\alpha < 0$, $\beta < 0$.

switching point at $t = \tau_2 > \tau_1$ to return to the origin. At most two switching points are admissible, so the third part must pass through the origin and it is easily seen that this part is the continuation of l to the right. The final time, the time at which we are back at the origin, is denoted by t_f.

The conclusion is that when $\alpha < 0$ and $\beta < 0$, in which case the ship is called stable, we can always go back to the origin in the (η, r) plane. Let us now return to the tree dimensional space (θ, v, r). The final position will be denoted by $(\gamma, 0, 0)$. If we choose τ_1, everything is determined

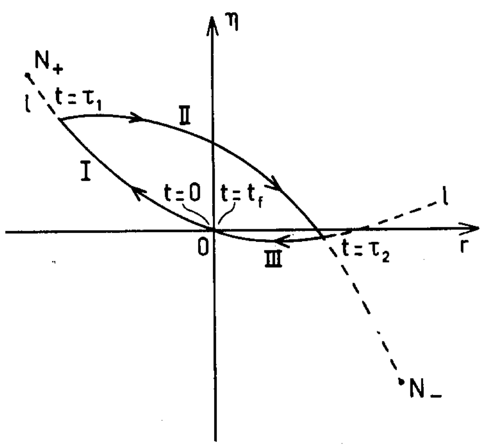

Figure 5.2 The three parts of the trajectory.

and γ is a unique and obviously continuous function of τ_1. If γ is given beforehand, the right value of τ_1 can be found by interpolation.

As will be proved in section 7 we have, in the case of an optimal trajectory, the remarkable formula:

$$\gamma = \pm Q\{\tau_1 - (\tau_2 - \tau_1) + (t_f - \tau_2)\}\,, \tag{5.1}$$

which is valid when the ship starts at $t = 0$ its change of course with $\delta = \pm\delta_{max}$. If one chooses the first switch of the rudder near N_+, τ_1 can be made arbitrarily large, while $(\tau_2 - \tau_1)$ and $(t_f - \tau_2)$, the times necessary for the ship to run along the second and third part of its trajectory, remain of the same order. So γ may be chosen arbitrarily; each change of course can be realized.

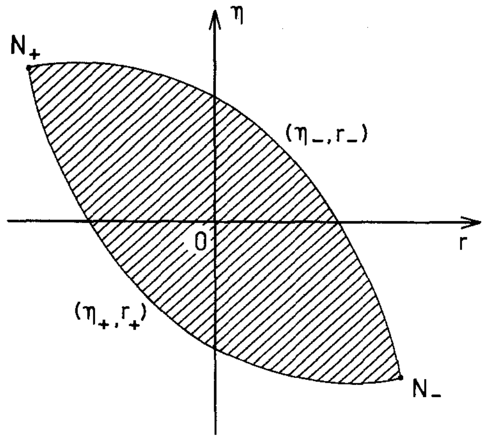

Figure 5.3 The attainable region of the ship in the (η, r) plane.

The shaded region in Figure 5.3 is the "attainable region" of the ship. It is impossible for the helmsman to give his ship a transverse velocity η and a rotational velocity r belonging to points outside the shaded area, when he starts from $\eta = r = 0$. This can be proved as follows. We use the theorem [4], [8] which states that every point in the (η, r) plane, which can be reached in one way or another from the origin, can also be reached by an optimal control. Hence a point which cannot be reached by an optimal control cannot be reached at all. However by optimal steering we cannot escape from the open region, which is bounded by the trajectories (η_+, r_+), passing through N_- and (η_-, r_-) passing through N_+. Hence we remain in the shaded region by any steering.

6. Unstable Ships; $\alpha > 0$, $\beta < 0$ (or $\alpha < 0$, $\beta > 0$).

In this case the solutions of equation (4.3) resemble hyperbolas [2]. These solutions are drawn in Figure 6.1.

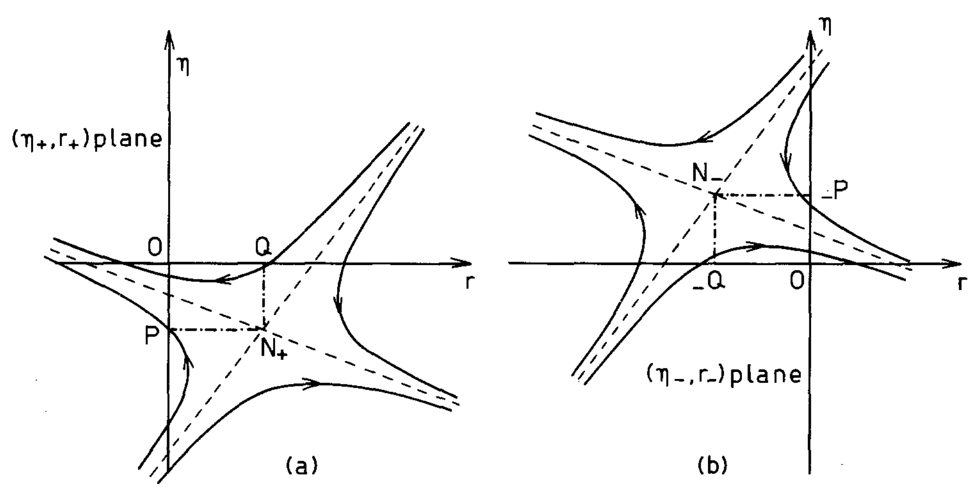

Figure 6.1 The trajectories $(\eta_\pm, r_\pm)$ in the (η, r) plane, in the case $\alpha > 0$, $\beta < 0$.

The solutions (η_+, r_+) have a saddlepoint at N_+, the solutions (η_-, r_-) have a saddlepoint at N_-. A difficulty arises in this case. Let us consider Figure 6.2. If we choose τ_1 arbitrarily, we cannot always return to the origin. If we take the first switch e.g. at point B, we cannot find a second switch, in such a way that the third part of the trajectory passes through the origin. Being at point B, it is impossible to return to the origin, by any steering. Suppose the contrary;

then also an optimal trajectory, which returns to the origin, exists and that is not possible. We speak of an unstable ship. Here we once more stress that the forward velocity is kept constant during the manoeuvre. Of course the linearized equations are no longer valid as the ship moves further away from the origin; that is the case when τ_1 is chosen too large. The shaded strip

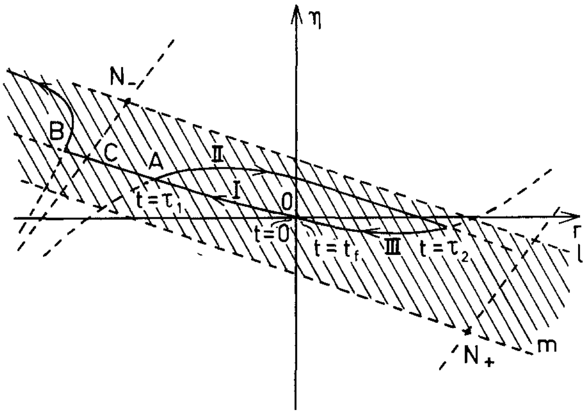

Figure 6.2 The possible optimal trajectories.

between the lines l and m is the attainable region. Each change of course can again be realized. The time interval $(\tau_2-\tau_1)$ can be made arbitrarily great in comparison with τ_1 and $(t_f-\tau_2)$, as we choose the first switch near, but to the right of point C. So by (5.1) each γ can be realized.

The case $\alpha<0$, $\beta>0$ can be treated similarly. It is the retrograde motion of Figure 6.2.

7. The Calculation of τ_2, t_f and $\theta(t_f)$, when τ_1 is given

We now will derive some formulas, which give τ_2, t_f and θ as a function of τ_1. These derivations are valid for stable as well as unstable ships. The equations of the first part of the trajectory are

$$\eta_+(t)=E_1 e^{\alpha t}+E_2 e^{\beta t}+P\,, \tag{7.1}$$

$$r_+(t)=M_1 E_1 e^{\alpha t}+M_2 E_2 e^{\beta t}+Q\,, \tag{7.2}$$

with M_1 and M_2 given by (4.4) and

$$E_1=\frac{Q-M_2\cdot P}{M_2-M_1}\,,\quad E_2=\frac{Q-M_1\cdot P}{M_1-M_2}\,. \tag{7.3}$$

At time $t=\tau_1$ the rudder switches. The values of $\eta_+(\tau_1)$ and $r_+(\tau_1)$ are known by equations (7.1) and (7.2). The equations of the second part are

$$\eta_-(t)=F_1 e^{\alpha t}+F_2 e^{\beta t}-P\,, \tag{7.4}$$

$$r_-(t)=M_1 F_1 e^{\alpha t}+M_2 F_2 e^{\beta t}-Q\,, \tag{7.5}$$

with

$$F_1=E_1(1-2e^{-\alpha\tau_1}),\quad F_2=E_2(1-2e^{-\beta\tau_1})\,. \tag{7.6}$$

The equations of the third part are the same as those from the first part within a translation with respect to t. Therefore the third part can be written as

$$\eta_+(t')=E_1 e^{\alpha t'}+E_2 e^{\beta t'}+P\,, \tag{7.7}$$

$$r_+(t')=M_1 E_1 e^{\alpha t'}+M_2 E_2 e^{\beta t'}+Q\,, \tag{7.8}$$

where $t'=t-t_f$, where the final time t_f is still an unknown constant. The trajectories (7.4), (7.5) and (7.7), (7.8) have two points of intersection: $(t=\tau_1, t'=\tau_1)$ and $(t=\tau_2, t'=\tau_4)$. Note that τ_4 will be negative. For the relevant point of intersection the following formulae are valid:

$$E_1 e^{\alpha\tau_4} + E_2 e^{\beta\tau_4} - F_1 e^{\alpha\tau_2} - F_2 e^{\beta\tau_2} + 2P = 0\,, \tag{7.9}$$

$$M_1 E_1 e^{\alpha\tau_4} + M_2 E_2 e^{\beta\tau_4} - M_1 F_1 e^{\alpha\tau_2} - M_2 F_2 e^{\beta\tau_2} + 2Q = 0\,. \tag{7.10}$$

Eliminating τ_2, we get

$$\frac{1}{\alpha} \ln \left\{ \frac{e^{\alpha\tau_4} - 2}{1 - 2e^{-\alpha\tau_1}} \right\} - \frac{1}{\beta} \ln \left\{ \frac{e^{\beta\tau_4} - 2}{1 - 2e^{-\beta\tau_1}} \right\} = 0\,. \tag{7.11}$$

From this formula we calculate τ_4 numerically by a Newton-Raphson technique. Now τ_4 is known and

$$\tau_2 = \frac{1}{\alpha} \ln \left\{ \frac{e^{\alpha\tau_4} - 2}{1 - 2e^{-\alpha\tau_1}} \right\}, \tag{7.12}$$

and $t_f = \tau_2 - \tau_4$.

Finally we calculate γ:

$$\gamma = \int_0^{\tau_1} r_+^{\mathrm{I}}\, dt + \int_{\tau_1}^{\tau_2} r_-^{\mathrm{II}}\, dt + \int_{\tau_2}^{t_f} r_+^{\mathrm{III}}\, dt\,, \tag{7.13}$$

where the indices I, II and III denote the parts of the trajectory. Taking together the first and the final term of the right hand side of equation (7.13) we obtain

$$\gamma = \int_{\tau_4}^{\tau_1} r_+\, dt + \int_{\tau_1}^{\tau_2} r_-\, dt = Q(2\tau_1 - \tau_2 - \tau_4)$$
$$= Q\{\tau_1 - (\tau_2 - \tau_1) + (t_f - \tau_2)\} \tag{7.14}$$

which is the result we already mentioned in equation (5.1).

If the ship started its change of course with $\delta = -\delta_{\max}$, the right hand side of equation (7.14) would have been provided with a minus sign.

For the unstable ships ($\alpha > 0$, $\beta < 0$) it can easily be found that the upperbound for the first switching time, in Figure 6.2 when the ship is at point C, is:

$$\tau_1 < \frac{\ln 2}{\alpha}\,. \tag{7.15}$$

8. Numerical Results

The calculations have been made with regard to three types of ships, (a), (b) and (c). The measured quantities of (a) and (b) are mentioned in [5] and [6] respectively. Type (c) has been derived from (b) by changing its coefficients somewhat.

coefficients	type of ship		
	(a)	(b)	(c)
b_3	−0.8946	−0.576	−0.5
b_4	−0.2856	−0.283	−0.3
b_5	+0.1453	+0.1795	+0.1795
c_3	−4.392	−3.886	−3.8
c_4	−2.719	−2.237	−2.2
c_5	−1.225	−1.307	−1.307

The eigenvalues α and β are calculated from these coefficients and are

α	−0.3623	−0.0688	+0.0147
β	−3.2515	−2.744	−2.715

We see that type (a) is stable, (b) is just stable and (c) is not stable. The results are shown in Figure 8.1. In this figure τ_1, τ_2 and t_f have been plotted against γ/δ_{max}, as Q, and therefore γ, is linear in δ_{max} by (4.4) and (7.14).

For small γ and small time intervals $[0, t_f]$ it is not real to admit jumps of the rudder as already has been remarked in section 4. In the next section this will be discussed in greater detail. Interpreting Figure 8.1, we will consider only those γ, of which $\gamma/\delta_{max} > 4$.

It is difficult to get type (a), the stable one, out of its original course, so τ_1 lasts long in comparison with the "correction" times $(\tau_2 - \tau_1)$ and $(t_f - \tau_2)$.

Type (c), the unstable one, changes its angular position very quickly. The "correction" time $(\tau_2 - \tau_1)$ is rather long. By (7.15) the upperbound for the first switching time is 47 units of time. For $0 \leqq \gamma/\delta_{max} \leqq 20$ the switch time τ_1 does not approach this limit.

Type (b), just stable, resembles type (c), but in this case τ_1 does not have an upperbound. So for large values of γ/δ_{max} the characters of type (b) and (c) in Figure 8.1 will differ considerably.

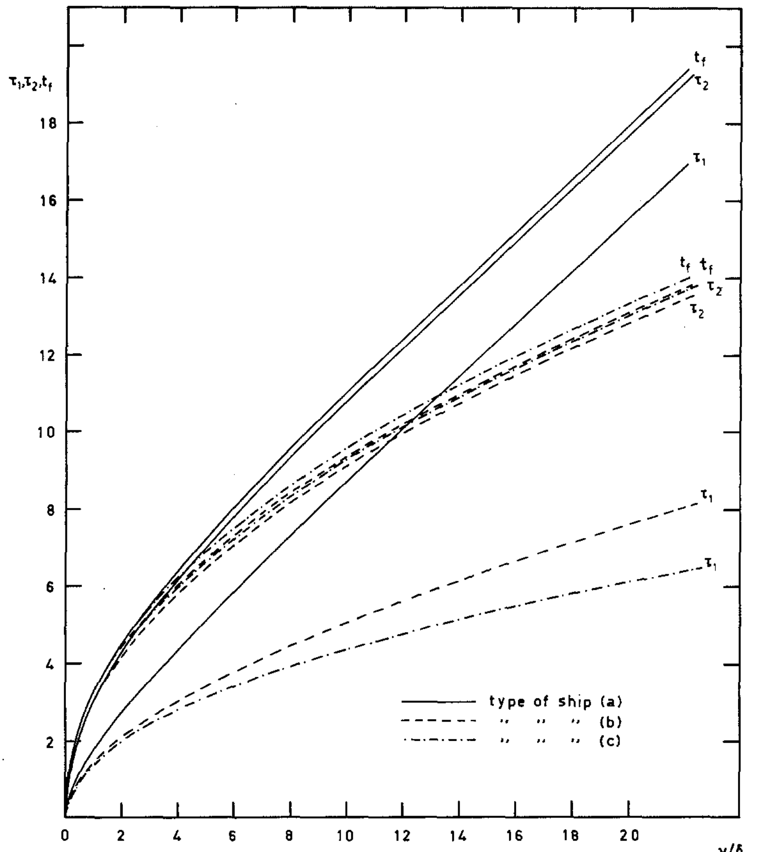

Figure 8.1

In Figures 8.2 and 8.3 the optimal trajectories of type (a) and (b) respectively, which belong to changes of course of $\pi/4$, $\pi/2$, $3\pi/4$ and π radials, in the physical horizontal plane, have been plotted. The coordinates x^* and y^* of the horizontal plane are fixed in space.

From section 2 it follows that

$$x^*(t) = \int_0^t \sin(\theta(t) + \operatorname{arctg}(\eta(t)))\,dt\,,$$

$$y^*(t) = \int_0^t \cos(\theta(t) + \operatorname{arctg}(\eta(t)))\,dt\,,$$

assuming that $v_0 = 1$ and that the process started at $t = 0$. Because $x^*(t)$ and $y^*(t)$ are not linear in δ_{max}, we have to make a choice for this quantity. In Figures 8.2 and 8.3 the maximum rudder

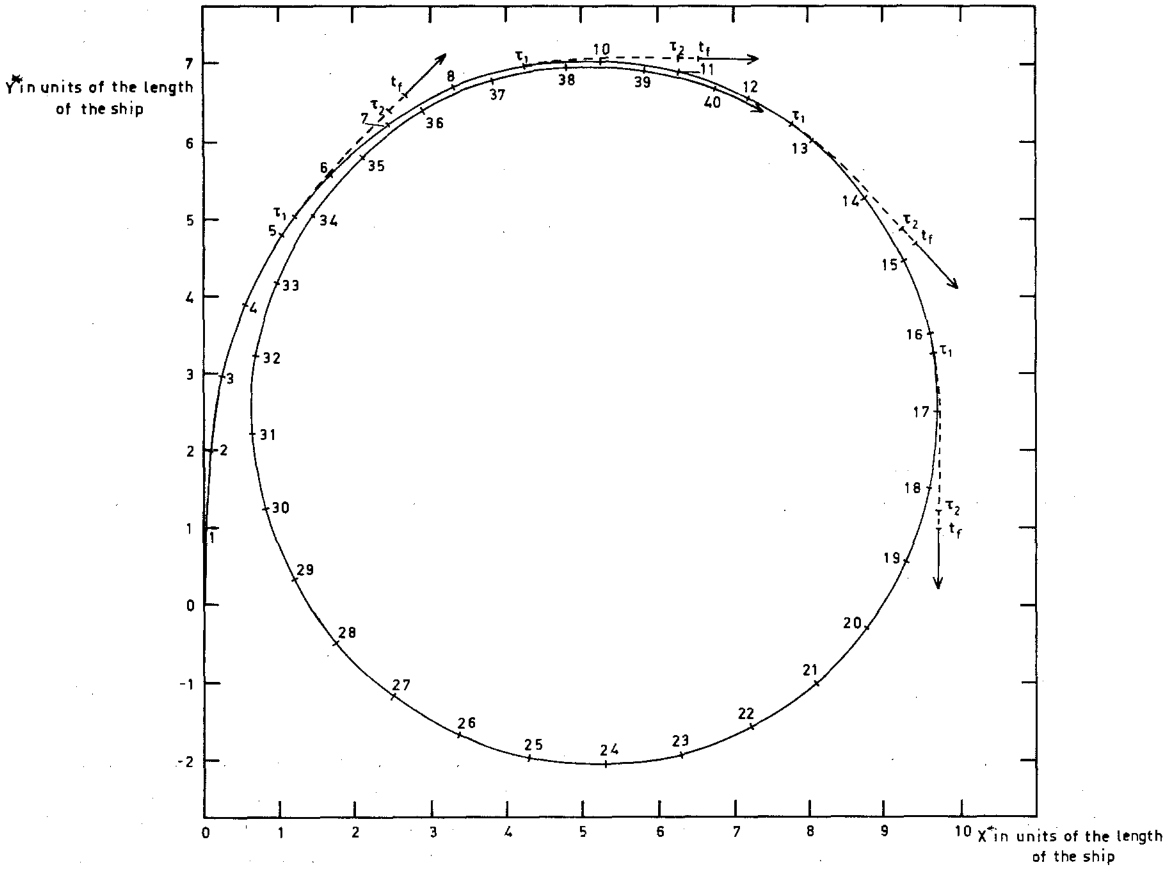

Figure 8.2 The optimal trajectories of type (a) in the horizontal plane, belonging to changes of course of $\pi/4$, $\pi/2$, $3\pi/4$ and π radials.

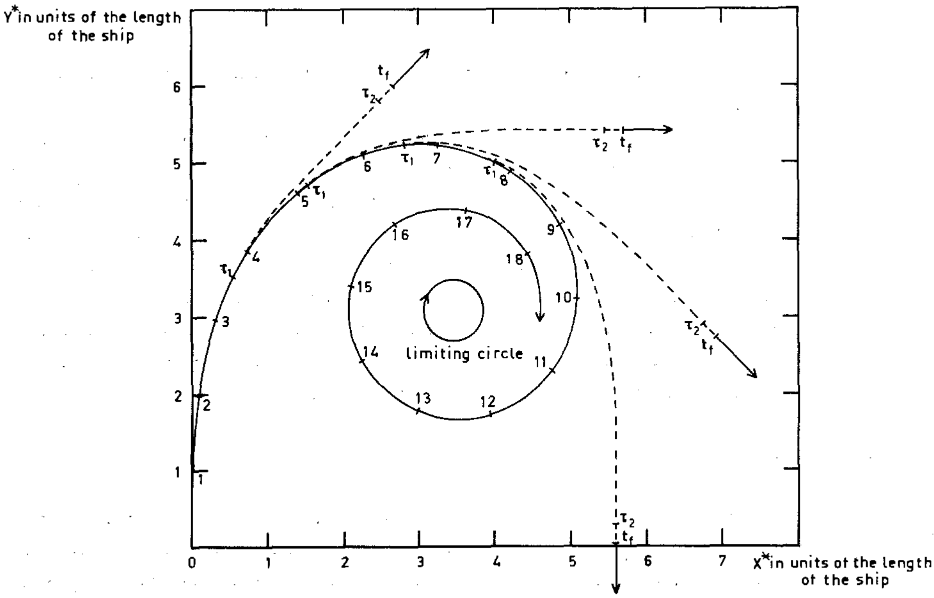

Figure 8.3 The optimal trajectories of type (b) in the horizontal plane, belonging to changes of course of $\pi/4$, $\pi/2$, $3\pi/4$ and π radials.

deviation δ_{max} was chosen to be 0.15 radials. The numbers along the trajectories denote the units of time. The radius of the smallest turning circle, the limiting circle, is $1/Q = 4{,}53$ for type (a) and 0.868 for type (b). The limiting circle of type (a) has not been drawn in the figure, because it nearly coincides with the plotted trajectory.

9. Comparison with the First Order Simulation of Nomoto

In [7] Nomoto gives a first order simulation of the steering problem. The equations do not contain the lateral velocity η:

$$\frac{d\theta}{dt} = r\,, \tag{9.1}$$

$$T\frac{dr}{dt} + r = K\cdot\delta\,, \tag{9.2}$$

T and K are constants.

Nomoto argues that K and T can be expressed in the constants b_3, b_4, b_5, c_3, c_4 and c_5 in the following way:

$$K = \frac{b_5c_3 - b_3c_5}{b_3c_4 - b_4c_3}\,, \tag{9.3}$$

$$T = -\left(\frac{b_3 + c_4}{b_3c_4 - b_4c_3} + \frac{c_5}{b_5c_3 - b_3c_5}\right). \tag{9.4}$$

Let us now consider again the problem of change of course by applying the maximum principle to system (9.1), (9.2). The same conditions as in the third order problem of section 4, are assumed: the process starts at $t=0$ at the point $r=0$, $\eta=0$ and ends at t_f with $r(t_f)=0$, $\eta(t_f)=0$, $\theta(t_f)=\gamma$. It follows from the theory of the maximum principle that at most one switch of the rudder exists. Choosing the switching time τ arbitrarily, $\tau > 0$, the time necessary for the change of course with angle γ, t_f can be calculated in the same way as discussed in the foreoging sections:

$$t_f = T\ln(2e^{\tau/T} - 1)\,, \tag{9.5}$$

$$\gamma = \theta(t_f) = \pm K\{\tau - (t_f - \tau)\}\,. \tag{9.6}$$

When the ship starts the change of course with $\delta = +\delta_{max}$, then the plus-sign is valid in equation (9.6), otherwise the minus-sign is.

The results of the calculations are shown in Figure 9.1. In Figure 9.2 the relative difference between the results of our method and Nomoto's have been plotted against γ/δ_{max}. If $\gamma/\delta_{max} > 4$ the theories agree within about 10% of the prediction of their final time t_f. When $\gamma/\delta_{max} < 4$, they differ appreciably. However both theories then become questionable, because the switching time will not be short with respect to the final time t_f. From Figure 8.1 we see that the intervals $(t_f - \tau_2)$ are small with respect to $(\tau_2 - \tau_1)$ and τ_1. This means that in essence we have two important steering periods. This is in agreement with the approximation of Nomoto, who uses a first order simulation and hence one switch. For this reason it is expected that in general Nomoto's approach will be sufficiently accurate for practical purposes.

10. Another Method to Solve the Steering Problem

The method used in section 4, 5 and 6 is practically limited to linear systems of third order. For systems of higher order it is in general very difficult to get a geometrical insight into the possible optimal trajectories. Therefore in this section another method of solution, which is also suitable for higher order systems of equations, will be discussed briefly. For a more complete treatment see [9].

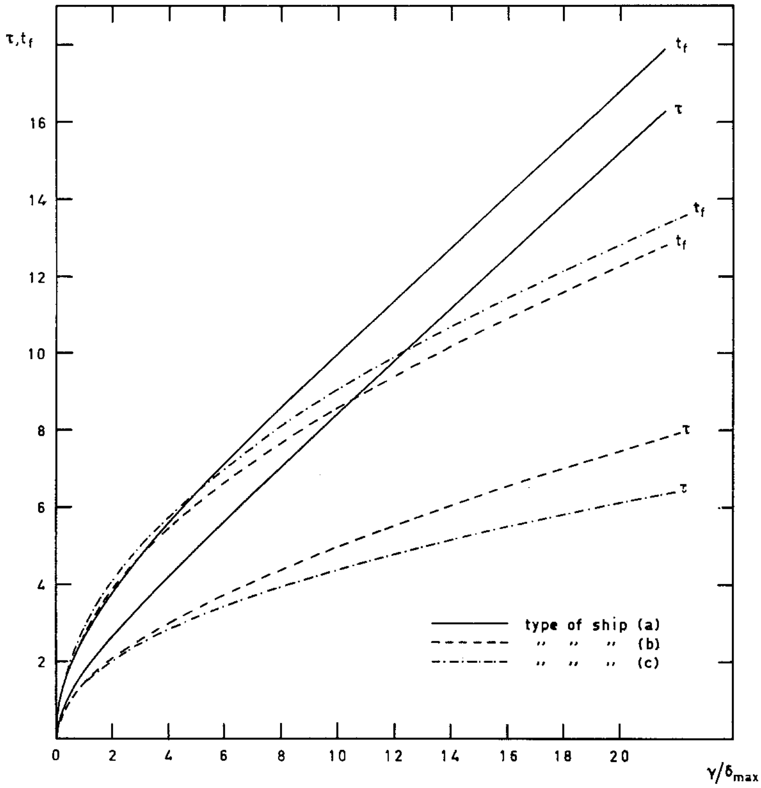

Figure 9.1

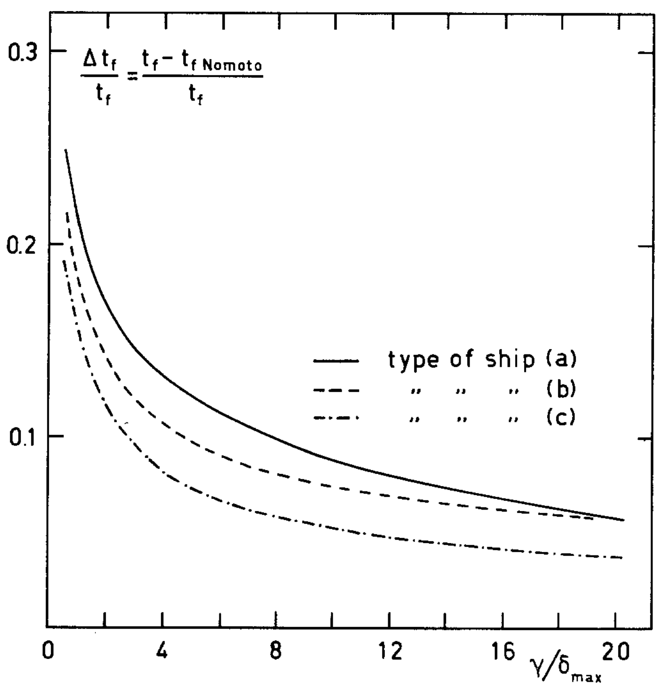

Figure 9.2

Equation (4.1) is considered again:

$$\frac{dx}{dt} = Ax + b\delta . \tag{10.1}$$

The maximum principle [8] states that the optimal control can be expressed explicitly in terms of solutions $\psi(t)$ of the related adjoint differential equation;

$$\delta(t) = \delta_{\max} \cdot \operatorname{sgn}(\psi(t), b) , \tag{10.2}$$

where $(\psi(t), b)$ denotes the innerproduct of $\psi(t)$ and b; $\psi(t)$ is a solution of

$$\frac{d\psi}{dt} = -A^T \psi , \tag{10.3}$$

where A^T is the transpose of A. If we choose an initial value for ψ, i.e. $\psi(t_0) = \psi_0$, the solution of (10.3) is known and $\delta(t)$ can be calculated from (10.2). However, the trajectory, which starts from x_0 at time t_0 and belongs to the calculated control function $\delta(t)$ will in general not pass through x_f. So the difficulty is how to choose ψ_0 so that the trajectory will pass through ψ_0. Neustadt [10] developed a method in which the difficulty of finding the right ψ_0 is reduced to the determination of the stationary value of a function, which contains the three components of ψ_0 as arguments. This value can be calculated numerically by using the gradient method of Powell [11].

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